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CHARACTERIZATION OF HILBERT SPACE MANIFOLDS REVISITED

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This exposition focuses not on manifolds modelled on Hilbert space but rather modelled on $s=(0,1)^{\infty}$, the product of countably many open intervals. The primary goal is to present an approach leading to the characterization of s-manifolds that relies exclusively on the geometry of s, contrasting with the original attack of Torunczyk in [21] that at times made essential use of the linear structure of Hilbert space. A secondary goal is to correct a misconception that has surfaced in several locations in characterizing manifolds in the non-locally compact setting; specifically, the usual notion of Z-set needs to be refined.

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Introduction

This exposition focuses not on manifolds modelled on Hilbert space but rather modelled on $s=(0,1)^{\infty}$, the product of countably many open intervals. The primary goal is to present an approach leading to the characterization of s-manifolds that relies exclusively on the geometry of s contrasting with the original attack of Torunczyk in [21] that at times made essential use of the linear structure of Hilbert space. A secondary goal is to correct a misconception that has surfaced in several locations in characterizing manifolds in the non-locally compact setting; specifically, the usual notion of Z-set needs to be refined.

A proof of the characterization of Hilbert cube manifolds appears in [13] together with an outline adapting the proof to the setting of Hilbert space manifolds. This

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strategy also avoids using the linear structure of Hilbert space and exhibits some similarities, as well as definite differences, with the strategy we put forth. The approach we adopt can be adapted to the setting of Hilbert cube manifolds; details can be found in [23].

Section 1 compares the concept of Z-set with that of strong Z-set. An example is presented that illustrates the difference between these concepts and is used to provide a counterexample to the statement [19]: if X is a complete separable ANR and $X-A$ is an s-manifold for a Z-set $A\subset X$, then X is an s-manifold. The form of the example is the model for other examples presented later in the paper that provide answers to questions that have circulated. The particular 'proof strategy' we employ for the characterization theorem focused our attention towards searching for examples exhibiting this form. (We have restricted our attention to manifolds modelled on s, which is homeomorphic to separable Hilbert space. The reader can find in [22] corrections that apply in the setting of Hilbert spaces of other weights.)

Section 2 presents a basic 'shrinking argument' that establishes that a fine homotopy equivalence $f: M^s \to X$ from an s-manifold to an ANR is a near homeomorphism provided the set of nondegenerate values of f is contained in a strong Z-set. A corollary is that if $A\subset X$ is a strong Z-set in a complete separable ANR X and $X-A$ is an s-manifold, then X is an s-manifold.

Section 3 refines the basic 'shrinking argument' in Section 2 to show that a fine homotopy equivalence $f: M^s \to X$ from an s-manifold to a complete separable ANR satisfying the discrete approximation property is a near homeomorphism provided the set of nondegenerate values of f is a countable union of Z-sets.

Section 4 contains a proof of Torunczyk's characterization of s-manifolds as topologically complete separable ANR's satisfying the *discrete approximation property* (*DAP*): for each countable family of maps $\alpha_i: I^\infty \to X$, $i=1,2,\ldots$, of Hilbert cubes to X and open cover $\mathcal{U}$ of X, there are $\mathcal{U}$-approximations $\beta_i: I^\infty \to X$, $i=1,2,\ldots$, such that the collection $\{\beta_i(I^\infty)\}_{i\geq 1}$ is discrete in X (i.e., each point in X has a neighborhood that meets at most one member of the collection). In Torunczyk's original proof (that used the incorrect result in [19]), the discrete approximation property was used only to show that the identity map $\mathrm{Id}: X\to X$ can be approximated, with cover close control, by a closed embedding $e: X\to X$ with $e(X)$ a Z-set. The example discussed in Section 1 is easily seen to satisfy this last property though it does not satisfy the discrete approximation property. The latter property is stronger as it forces Z-sets to be strong Z-sets (see Proposition 1.3).

Section 5 presents an example of a topologically complete separable ANR X that is not an s-manifold even though it satisfies, for each $n\geq 1$, the *discrete n-cells property*: for each countable family of maps $\alpha_i: I^n \to X$, $i=1,2,\ldots$, of n-cells to X and open cover $\mathcal{U}$ of X, there are $\mathcal{U}$-approximations $\beta_i: I^n\to X$, $i=1,2,\ldots$, such that the collection $\{\beta_i(I^n)\}_{i\geq 1}$ is discrete in X.

Section 6 sets forth a further refinement of this last example that eliminates a 'natural' extension of a homological characterization of Hilbert cube manifolds as in [12] to s-manifolds. The examples in Sections 5 and 6 as well as those in Section

1 cannot be found amongst ANR's X that arise as $X = Y - F$ where Y is a locally compact ANR and F is a countable union of Z-sets in Y (the reader is referred to [8] for details).

Section 7 is devoted to a brief discussion of the effect that the existence of the examples as in Section 1 has on characterizations of manifolds modelled on certain incomplete infinite dimensional spaces. (Namely, $\sigma = \{(x_i) \in s: x_i = 0$ for all but finitely many $i\}$ and $\Sigma = \{((x_i)_j) \in s^\infty: (x_i)_j = (0)$ for all but finitely many $j\}$).

An appendix contains a discussion of two properties of s-manifolds that are of central importance in the proof of the characterization theorem.

Terminology and notation

Spaces are separable and metric. A space is *topologically complete* provided it possesses a complete metric or, equivalently, it is a G_δ subset of any metric space in which it is embedded. A *fine homotopy equivalence* is a map $f: X \to Y$ such that, for each open cover $\mathcal{U}$ or Y, there is a map $g: Y \to X$ with gf $f^{-1}(\mathcal{U})$-homotopic to Id_X and fg $\mathcal{U}$-homotopic to Id_Y. Be warned that, in the setting of non-locally compact spaces, fine homotopy equivalences generally are not proper maps and are not onto (though they always have dense images). A map $f: X \to Y$ is a *near homeomorphism* provided for each open cover $\mathcal{U}$ of Y there is a homeomorphism $h: X \to Y$ $\mathcal{U}$-close to f. Additional terminology will be introduced as it is needed.

1. Z-sets versus strong Z-sets

The notion of negligibility encompassed in the commonly used term Z-set has undergone a variety of reformulations since its introduction. For a closed subset A of an ANR X, anyone of the following equivalent statements can be taken as the definition of *A is a Z-set in X*.

(a) For each open cover $\mathcal{U}$ of X, there is a map $\alpha: X \to (X - A)$ $\mathcal{U}$-close to Id_X.

(b) For each map $\alpha: I^\infty \to X$ and each $\varepsilon > 0$, there is a map $\beta: I^\infty \to (X - A)$ ε-close to α.

(c) For each positive integer n, each map $\alpha: I^n \to X$, and each $\varepsilon > 0$, there is a map $\beta: I^n \to (X - A)$ ε-close to α.

(d) For each space Y, each map $\alpha: Y \to X$, and each open cover $\mathcal{U}$ of X, there is a map $\beta: Y \to (X - A)$ $\mathcal{U}$-close to α.

If X happens to be locally compact as well, then the list can be expanded to include:

(*) For each open cover $\mathcal{U}$ of X, there is an open set $V \supset A$ and a map $\alpha: X \to (X - V)$ $\mathcal{U}$-close to the identity.

Unfortunately, a misconception held widely (including by all four of the authors for varying periods of time) was that in general the condition in (*) belonged in the list of equivalent formulations of Z-set. The following simple example shows that such is not the case.

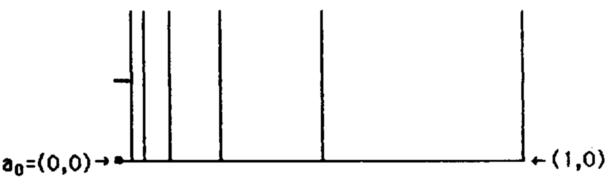
Key example. The example is the subset of the plane illustrated in Fig. 1; namely, $C=([0,1]\times\{0\})\cup(\bigcup_{n\geq 1}\{1/n\}\times[0,1])$.

The space C is a topologically complete AR and is locally compact at each point other than $a_0=(0,0)$, and the point a_0 is a Z-set but does not satisfy (*). For each $t\in[0,1)$, a map $h_t: C\to C$ is determining by requiring that $h_t(1,0)=(1,0)$, $h_t(1/n,1)=(1/n,1)$, h_t maps $[0,1]\times\{0\}$ linearly onto $[t,1]\times\{0\}$, h_t maps $\{1/n\}\times[t,1]$ linearly onto $\{1/n\}\times[0,1]$, and h_t maps $\{1/n\}\times[0,t]$ linearly onto the subinterval of $[0,1]\times\{0\}$ determined by $h_t(1/n,t)=(1/n,0)$ and $h_t(1/n,0)=(t+(1-t)/n,0)$. The maps $\{h_t\}_{0\leq t<1}$ form an instantaneous deformation of C into $C-a_0$; that is, $h_0=\mathrm{Id}_C$ and $h_t(C)\subset(C-a_0)$ for $0<t<1$. It follows that $\{a_0\}$ is a Z-set, but it cannot satisfy (*) as $C-V$ is disconnected for all (small) neighborhoods V of a_0.

Henceforth, we shall reserve the term *strong Z-set* for closed subsets A of an ANR X that satisfy (*).

Before using the space C to produce a counterexample to [19; Proposition 5.1] (the map defined by the formula at the bottom of p. 104 is not in general continuous), we need to introduce additional notation. For a continuous map $f: X\to Y$ and a closed subset $A\subset Y$, the adjunction space $X\cup_f A$ is defined to be the disjoint union $(X-f^{-1}(A))\cup A$ with the topology consisting of the usual open subsets of $X-f^{-1}(A)$ together with sets of the form $f^{-1}(U-A)\cup(U\cap A)$ for open subsets $U\subset Y$. There is an 'induced' factorization $f=p_Af_A$ where $f_A: X\to X\cup_f A$ is defined by setting $f_A=\mathrm{Id}_{X-f^{-1}(A)}$ on $X-f^{-1}(A)$ and $f_A=f$ on $f^{-1}(A)$ and where $p_A: X\cup_f A\to Y$ is defined by setting $p_A=f$ on $X-f^{-1}(A)$ and $p_A=\mathrm{Id}_A$ on A. In an important special case of a projection $\pi: Y\times Z\to Y$, we shall denote the adjunction space $(Y\times Z)\cup_\pi A$ by $(Y\times Z)_A$ for a closed subset $A\subset Y$ (and refer to it as a *reduced product*). For our purposes, we shall only be interested in fine homotopy equivalences $f: X\to Y$ between topologically complete ANR's and in Z-sets $A\subset Y$. In this case, $X\cup_f A$ is a topologically complete ANR containing A as a Z-set and both f_A and p_A are fine homotopy equivalences.

Returning to the 'key example' C, form the reduced product $(C\times s)_{a_0}$. Now, the point $a_0\in(C\times s)_{a_0}$ is a Z-set as $a_0\in C$ is a Z-set, but it is not a strong Z-set for that would imply easily that $a_0\in C$ is a strong Z-set (a more general statement appears in Corollary 1.2). Consequently, $(C\times s)_{a_0}$ is not an s-manifold (as s-manifolds possess the strong discrete approximation property that, as revealed in



C=([0,1]x{0})∪(∪{{1/n}x[0,1]:n≥1})

Fig. 1.

Proposition 1.3, implies that Z-sets are strong Z-sets). However, $(C \times s)_{a_0} - \{a_0\} = (C - a_0) \times s$ is an s-manifold (see Corollary 4.1). The specifics of the example C are immaterial: for any topologically complete ANR X and closed subset $A \subset X$ that is a Z-set but not a strong Z-set, $A \subset (X \times s)_A$ is a Z-set and $(X \times s)_A - A$ is an s-manifold, but $(X \times s)_A$ is not an s-manifold.

Before proceeding to the next section that contains a correct version of [19; Proposition 5.1], we state results about strong Z-sets that we shall need. Finding proofs is left as an exercise; and/or the reader may consult [6; Section 1] where general properties of strong Z-sets are set forth.

Proposition 1.1. *Let $f: X \to Y$ be a fine homotopy equivalence between ANR's. A closed subset $A \subset Y$ is a strong Z-set if and only if $A \subset X \cup_f A$ is a strong Z-set.*

Corollary 1.2. *A closed subset A of an ANR X is a strong Z-set if and only if A is a strong Z-set in $(X \times s)_A$.*

Proposition 1.3. *If a topologically complete ANR X satisfies the discrete approximation property, then every Z-set in X is a strong Z-set in X.*

In fact, the preceding proposition follows from the next result. Its content is a useful detection principle that is analogous to Condition (b) in the list, at the beginning of the section, of equivalent formulations of the notion of Z-set.

Proposition 1.4. *A closed subset A of ANR X is a strong Z-set if and only if, for each open cover $\mathcal{U}$ of X and sequence of maps $\alpha_1, \alpha_2, \ldots$ of I^∞ to X, there are $\mathcal{U}$-approximations $\beta_1, \beta_2, \ldots$ such that $\bigcup \{\beta_i(I^\infty): 1 \leq i < \infty\}$ misses a neighborhood of A.*

2. Strong Z-set shrinking

The setting is a fine homotopy equivalence $f: M^s \to X$ from an s-manifold to an ANR. In this section as well as the next, conditions are imposed on the non-degeneracy of f that are shown, using a shrinking argument, to imply that f is a near homeomorphism. The results of these two sections are combined in Section 4 to establish the characterization theorem.

For a map $f: X \to Y$ between topologically complete spaces, a point $y \in Y$ is called a *nondegenerate value* of f provided, for a complete metric ρ on X, there is an $\varepsilon > 0$ such that the ρ-diameter of $f^{-1}(U)$ is greater than ε for each neighborhood U of y. It is easily seen that different choices of complete metrics for X yield the same nondegenerate values. (When $f(X)$ is dense in Y, a metric independent determination of a nondegenerate value y is that either $f^{-1}(y) = \emptyset$, or $f^{-1}(y)$ contains at least two points, or $f^{-1}(y) = \{x\}$ but $f^{-1}(\mathcal{B})$ is not a neighborhood basis for x

where $\mathcal{B}$ is a neighborhood basis for y.) The set of nondegenerate values of f is denoted by N_f and is an F_σ-subset of Y. Furthermore, the restriction of f is a homeomorphism from $f^{-1}(Y-N_f)$ to $Y-N_f$. In case that f is proper, a rare occurrence in the setting of s-manifolds, $N_f=\{y: f^{-1}(y)\neq \text{point}\}$. Perhaps it is worth emphasizing that, in general, fine homotopy equivalences are not onto, though they have dense images, and points not in the image are necessarily nondegenerate values!

The shrinking criterion that we shall use to detect that maps are near homeomorphisms is the following (see [21]). A map $f: X\to Y$ between topologically complete spaces is a near homeomorphism provided, for each pair of open covers $\mathcal{V}$ of X and $\mathcal{U}$ of Y, there is a homeomorphism $h: X\to X$ such that fh and f are $\mathcal{U}$-close and each point $y\in Y$ has a neighborhood W_y such that $h(f^{-1}(W_y))$ is contained in an element of $\mathcal{V}$.

We are ready to state the Strong Z-Set Shrinking Theorem. Its proof relies on the proposition that follows it. In comparing the statements of the theorem and proposition, the appropriate point of view is that the latter is 'measuring' the nondegeneracy in the domain while the former is 'measuring' it in the range.

Theorem 2.1 (Strong Z-set shrinking). *If $f: M^s\to X$ is a fine homotopy equivalence from an s-manifold to a topologically complete ANR and* $\mathrm{cl}(N_f)$ *is a strong Z-set in X, then f is a near homeomorphism.*

The proof will be given following that of the next result that supplies the essential ingredient for 'shrinking'.

Proposition 2.2. *If a topologically complete ANR X is expressed as the union $X=M\cup A$ of an s-manifold M and a strong Z-set A, then the inclusion $M\subset X$ is a near homeomorphism.*

Proof. Let $\mathcal{U}$ be an open cover of X and $\mathcal{V}$ an open cover of M. Since A is a Z-set, there is a map $g: X\to M$ that is $\mathcal{U}$-homotopic to the Id_X. Name open covers $\mathcal{U}_0$ of X and $\mathcal{V}_0$ of M such that $\mathcal{U}_0$ refines both $\mathcal{U}$ and $g^{-1}(\mathcal{V})$, and $\mathcal{V}_0$ refines both $\mathcal{V}$ and $\mathcal{U}_0|_M=\{U\cap M: U\in\mathcal{U}_0\}$. Specify an additional open cover $\mathcal{V}_1$ of M that star refines $\mathcal{V}_0$ and a homeomorphism $H: M\times I\to M$ that is $\mathcal{V}_1$ close to projection (see Appendix). Note that any map of the form HhH^{-1}, where $h: M\times I\to M\times I$ preserves M-coordinates, is $\mathcal{V}_0$-close to Id_M.

The 'shrinking' homeomorphism arises as a composition $h=h_2h_3h_1^{-1}$ of homeomorphism of M that we shall now construct.

Construction of h_1. Let $\mathcal{U}_1$ be an open star refinement of $\mathcal{U}_0$. There exists an instantaneous deformation $\{\alpha_t: 0\leq t\leq 1\}$ of X into M with the 'tracks' of the deformation refining $\mathcal{U}_1$ and with $\alpha_1(X)$ disjoint from a neighborhood of A [6; Corollary 1.2]. Choose a closed Z-embedding $e: H(M\times\{0\})\to M$ such that e is $\mathcal{U}_1|_M$-homotopic to $\alpha_1|_{H(M\times\{0\})}$ [21] and such that the image of e is disjoint from a neighborhood of A. Since e is $\mathcal{U}_0|_M$-homotopic to the inclusion of $H(M\times\{0\})$ into

M, an application of Z-set unknotting (see Appendix) gives a homeomorphism $h_1: M \to M$ extending e that is $\mathcal{U}_0|_M$-homotopic to Id_M.

Construction of h_2. Choose a closed Z-embedding $e_0: H(M \times \{1\}) \to M$ $\mathcal{V}_0$-close to the restriction $gih_1|_{H(M\times\{1\})}$ (where $i: M \to X$ is the inclusion) and extend it to a homeomorphism $h_2: M \to M$. Since gih_1 is $\mathrm{st}(\mathcal{U}_0|_M, \mathcal{U}|_M)$-homotopic to Id_M, we arrange that h_2 is $\mathrm{st}^2(\mathcal{U}|_M)$close to Id_M.

Observation. For each $a \in A$, if W is a neighborhood of a in X that is contained in an element of $\mathcal{U}_0$ and $p: M \times I \to M \times \{1\}$ is projection, then $h_2HpH^{-1}h_1^{-1}(W-A)$ is contained in an element of $\mathrm{st}^4(\mathcal{V})$.

In order to verify this, notice that, as h_2 is $\mathcal{V}_0$-close to gih_1 on $H(M \times \{1\})$, it suffices to show that $gih_1HpH^{-1}h_1^{-1}(W-A)$ is contained in an element of $\mathrm{st}^3(\mathcal{V})$. Since $\mathcal{U}_0$ refines $g^{-1}(\mathcal{V})$, it suffices to show that $ih_1HpH^{-1}h_1^{-1}(W-A)$ is contained in an element of $\mathrm{st}^3(\mathcal{U}_0)$. The last containment is clear since $W-A$ is contained in an element of $\mathcal{U}_0$, h_1 is $\mathcal{U}_0|_M$-close to Id_M, $\mathcal{V}_0$ refines $\mathcal{U}_0|_M$, and HpH^{-1} is $\mathcal{V}_0$-close to Id_M.

Construction of h_3. Since A and $h_1(H(M \times \{0\}))$ are disjoint closed subsets of X, A has a closed neighborhood N missing $h_1(H(M \times \{0\}))$. Consequently, $H^{-1}h_1^{-1}(N-A)$ is a closed subset of $M \times I$ and is contained in $M \times (0, 1]$. Let $h_0: M \times I \to M \times I$ be a homeomorphism that preserves M-coordinates and 'pushes' $H^{-1}h_1^{-1}(N-A)$ near $M \times \{1\}$. For an appropriately chosen h_0, the homeomorphism $h_3 = Hh_0H^{-1}$ is 'close enough' to HpH^{-1} on neighborhoods $h_1^{-1}(W-A)$ (where $W \subset N$) as in the above observation that $h_2h_3h_1^{-1}(W-A)$ is contained in an element of $\mathrm{st}^4(\mathcal{V})$. □

Proof of Theorem 2.1. The 'trick', performed in the next paragraph, is to approximate f by a fine homotopy equivalence $g: M^s \to X$ such that N_g is contained in $\mathrm{cl}(N_f)$ and $g^{-1}(\mathrm{cl}(N_g))$ is a (possibly empty) Z-set, necessarily a strong Z-set as M^s is an s-manifold. Proposition 2.1 applies to show that both inclusions $i: g^{-1}(X-\mathrm{cl}(N_g)) \to M^s$ and $j: (X-\mathrm{cl}(N_g)) \to X$ are near homeomorphisms. Therefore, g is approximable by a homeomorphism of the form $h_2g_0h_1^{-1}$ where h_1 approximates i, h_2 approximates j, and $g_0 = g|g^{-1}(X-\mathrm{cl}(N_g))$.

The fine homotopy equivalence g approximating f is the limit of a sequence $f_0, f_1, \ldots$ of fine homotopy equivalences specified recursively. Start by naming a countable set of embeddings $e_1, e_2, \ldots$ of the Hilbert cube I^∞ into M^s, having pairwise disjoint images, that is dense in the space of maps of I^∞ to M^s. Set $f_0 = f$. The map $f_0e_1: I^\infty \to X$ is homotopic, by a 'small' homotopy, to a Z-embedding $j_1: I^\infty \to X-\mathrm{cl}(N_{f_0})$. Using an approximate 'lift' of the homotopy as a guide, Z-set unknotting produces a homeomorphism $h_1: M^s \to M^s$, fixed outside a small neighborhood of $f_0^{-1}f_0(e_1(I^\infty))$, such that $h_1e_1 = f_0^{-1}j_1$ and f_0h_1 is 'close to' f_0. Set $f_1 = f_0h_1$. Observe that f_1 is one to one over the image of $e_1(I^\infty)$ (that is, $f_1^{-1}f_1(m) = m$ for each $m \in e_1(I^\infty)$). In recursive fashion, the sequence $f_0, f_1, f_2, \ldots$ is constructed so that $f_{i+1} = f_ih_{i+1}$ where $h_{i+1}: M^s \to M^s$ is a homeomorphism fixed outside a small neighborhood of $f_i^{-1}f_i(e_{i+1}(I^\infty))$ missing $\bigcup\{e_k(I^\infty): 1 \le k \le i\}$. Further, letting N_{i+1}

denote the nondegeneracy set of f_{i+1}, we want $h_{i+1}e_{i+1} = f_i^{-1}j_{i+1}$ where $j_{i+1}: I^\infty \to X - \mathrm{cl}(N_i)$ is an embedding approximating $f_i e_{i+1}$, and we want $f_i h_{i+1}$ to be 'close to' f_i. Note that $\bigcup\{f_{i+1}(e_k(I^\infty)): 1 \leq k \leq i+1\} \subset X - \mathrm{cl}(N_{i+1})$. Since $N_i = N_f$ (as is easily seen), if sufficient care is exercised in specifying the 'closeness' of f_{i+1} to f_i, the map $g = \lim\{f_i\}$ will be a fine homotopy equivalence approximating f with $N_g \subset \mathrm{cl}(N_f)$ and $g(e_i(I^\infty)) \subset X - \mathrm{cl}(N_f)$ for each i. It follows easily that $g^{-1}(\mathrm{cl}(N_g))$ is a Z-set in M^s.

3. σ-Z-set shrinking

Once again the setting is a fine homotopy equivalence $f: M^s \to X$ from an s-manifold to an ANR. The goal in this section is to use essentially the same argument as in the preceding section to show that if N_f is a σ-Z-set (i.e., a countable union of Z-sets) and X satisfies the discrete approximation property, then f is a near homeomorphism. In spaces that satisfy the discrete approximation property, Z-sets are strong Z-sets; a proof can be found in [6; Section 1]. Perhaps it is worth pointing out that merely assuming N_f is a strong σ-Z-set does not come close to forcing X to be of the 'right' type to be an s-manifold; for example, the inclusion $s \to I^\infty$ is a fine homotopy equivalence whose nondegeneracy set is the σ-Z-set $I^\infty - s$. Even more intimidating are examples of the type found in [4] which also contain an embedded copy of s whose complement is a strong σ-Z-set.

Theorem 3.1 (σ-Z-set shrinking). *If $f: M^s \to X$ is a fine homotopy equivalence from an s-manifold to a topologically complete ANR satisfying the discrete approximation property and N_f is a σ-Z-set, then f is a near homeomorphism.*

The proof of Theorem 3.1 parallels closely that of Theorem 2.1, that is, an approximating fine homotopy equivalence g will be produced that is seen to be a near homeomorphism using the next proposition in place of Proposition 2.2.

Proposition 3.2. *If $f: M^s \to X$ is a fine homotopy equivalence from an s-manifold to a topologically complete ANR satisfying the discrete approximating property where $f^{-1}(N_f)$ is a (possibly empty) σ-Z-set in M and N_f is a σ-Z-set in X, then f is a near homeomorphism.*

Proof. Let $\mathcal{U}$ be an open cover of X and $\mathcal{V}$ an open cover of M. Specify a map $g: X \to M$ such that gf is $f^{-1}(\mathcal{U})$-homotopic to Id_M. Name open covers $\mathcal{U}_0$ of X and $\mathcal{V}_0$ of M such that $\mathcal{U}_0$ refines both $\mathcal{U}$ and $g^{-1}(\mathcal{V})$ and such that $\mathcal{V}_0$ refines both $\mathcal{V}$ and $f^{-1}(\mathcal{U}_0)$. Specify an additional open cover $\mathcal{V}_1$ of M that star refines $\mathcal{V}_0$ and a homeomorphism $H: M \times I \to M$ that is $\mathcal{V}_1$-close to projection. Note that any homeomorphism of the form HhH^{-1}, where $h: M \times I \to M \times I$ preserves M-coordinates, is $\mathcal{V}_0$-close to Id_M.

The shrinking homeomorphism arises as a composition $h = h_2 h_3 h_1^{-1}$ of homeomorphisms of M that we shall now construct.

Construction of h_1. We shall need a closed Z-embedding $e: H(M\times\{0\}) \to M$ whose image misses an f-saturated neighborhood of $f^{-1}(N_f)$. (Since the maps we encounter generally are not onto, we emphasize that by an *f-saturated neighborhood* of a set $f^{-1}(A)$, for $A \subset X$, is meant a neighborhood of the form $f^{-1}(N)$ for some neighborhood N of A.) Deducing that there is such a map is based on Baire properties of the function space $\mathscr{C}(H(M\times\{0\}), X)$. Following [21; Section 1] where the reader can find additional details, $\mathscr{C}(Y, X)$ denotes the space of continuous functions endowed with the limitation topology. In the case that X is metrizable and topologically complete, $\mathscr{C}(Y, X)$ is a Baire space (i.e., the intersection of countably many dense G_δ subsets is dense). Asserted in [21; Lemma 3.8] is that the subspace of $\mathscr{C}(H(M\times\{0\}), X)$ consisting of closed Z-embeddings is a dense G_δ and it is easily seen that the subspace of maps whose image misses a neighborhood of N_f is likewise a dense G_δ. If e_0 is chosen to be in the intersection of these two G_δ's, then $e = f^{-1}e_0$ is a closed Z-embedding whose image misses a saturated neighborhood of $f^{-1}(N_f)$. Extend e to a homeomorphism $h_1: M \to M$. By choosing e_0 close to $f|_{H(M\times\{0\})}$, we arrange that h_1 is $f^{-1}(\mathscr{U}_0)$-homotopic to Id_M (see the construction of h_1 in the proof of Proposition 2.2).

Construction of h_2. Choose a closed Z-embedding $e_1: H(M\times\{1\}) \to M$ $\mathscr{V}_0$-homotopic to the restriction of $gfh_1|_{H(M\times\{1\})}$ and extend it to a homeomorphism $h_2: M \to M$. Since gfh_1 is $\mathrm{st}(f^{-1}(\mathscr{U}_0), f^{-1}(\mathscr{U}))$-homotopic to Id_M, we can arrange that h_2 is $\mathrm{st}^2(f^{-1}(\mathscr{U}))$ close to Id_M.

Observation. For each $a \in N_f$, if W is a neighborhood of a that is contained in an element of $\mathscr{U}_0$ and $p: M\times I \to M\times\{1\}$ is projection, then $h_2HpH^{-1}h_1^{-1}(f^{-1}(W))$ is contained in an element of $\mathrm{st}^4(\mathscr{V})$.

In order to verify this, notice that, as h_2 is $\mathscr{V}_0$-close to gfh_1 on $H(M\times\{1\})$, it suffices to show that $gfh_1HpH^{-1}h_1^{-1}(f^{-1}W))$ is contained in an element of $\mathrm{st}^3(\mathscr{V})$. Since $\mathscr{U}_0$ refines $g^{-1}(\mathscr{V})$, it suffices to show that $h_1HpH^{-1}h_1^{-1}(f^{-1}(W))$ is contained in an element of $\mathrm{st}^3(f^{-1}(\mathscr{U}_0))$. The last containment is clear since W is contained in an element of $\mathscr{U}_0$, h_1 is $f^{-1}(\mathscr{U}_0)$-close to Id_M, $\mathscr{V}_0$ refines $f^{-1}(\mathscr{U}_0)$, and HpH^{-1} is $\mathscr{V}_0$-close to Id_M.

Construction of h_3. Up to this point, the proof has mimicked that of Proposition 2.2 needing only minor deviations and, while the construction of h_3 requires substantially more 'technical' care, it too arises as $h_3 = Hh_0H^{-1}$ where h_0 preserves M-coordinates. Since $M\times\{0\}$ misses the preimage under $(h_1H)^{-1}$ of an f-saturated neighborhood of $f^{-1}(N_f)$, for any given point $a \in N_f$, we could choose a neighborhood W_a of a contained in an element of $\mathscr{U}_0$ with $\mathrm{cl}(f^{-1}(W_a)) \cap h_1H(M\times\{0\}) = \emptyset$ and specify that h_0 'push' $(h_1H)^{-1}(f^{-1}(W_a))$ so near $M\times\{1\}$ that $h_2Hh_0H^{-1}h_1^{-1}$ 'shrinks' $f^{-1}(W_a)$. The problem is that presumably any preimage under $(h_1H)^{-1}$ of an f-saturated neighborhood of $f^{-1}(N_f)$ is dense, thereby making it impossible to simultaneously do this for every point of N_f. The saving observation is that, for any $a \in N_f$ sufficiently close to $fh_1H(M\times\{0\})$, $f^{-1}(a)$ has an f-saturated neighborhood having diameter close to zero. We now proceed with the construction of h_0; as before, h_0 is the identity on $M\times\{0\}$ and moves only M-coordinates.

For a pair of maps $\varepsilon, \delta: M\to[0,1]$ with $\varepsilon(m)\leq\delta(m)$ for $m\in M$, set $\Gamma(\varepsilon,\delta)=\{(m,t): \varepsilon(m)\leq t\leq\delta(m)\}$ and set $p_{\varepsilon,\delta}: M\times I\to\Gamma(\varepsilon,\delta)$ equal to the retraction sending $\{m\}\times[0,\varepsilon(m)]$ to $(m,\varepsilon(m))$ and $\{m\}\times[\delta(m),1]$ to $(m,\delta(m))$.

Guided by the observation following the construction of h_2, we find a map $\delta: M\to(0,1)$ such that

(i) each $a\in N_f$ has a neighborhood W_a such that $h_2Hp_{1-\delta,1}H^{-1}h_1^{-1}(f^{-1}(W_a))$ is $\mathrm{st}^4(\mathcal{V})$-small, and

(ii) for each $(m,t)\in M\times I$, $h_2H(B_{2\delta(m)}(m)\times B_{2\delta(m)}(t))$ is $\mathrm{st}^4(\mathcal{V})$-small (where, generally $B_\varepsilon(x)$ denotes the ε-ball about x).

Next, we shall recursively specify maps $\varepsilon(i): M\to(0,1]$, $i=0,1,\dots,$ with $\varepsilon(0)(m)=1$ and $\varepsilon(i)(m)>\varepsilon(i+1)(m)$ for each $m\in M$ so that every point $a\in N_f$ has a neighborhood W_a such that either

$(\mathrm{iii})_1$ $H^{-1}h_1^{-1}(f^{-1}(W_a))\subset\Gamma_{\varepsilon(i+2),\varepsilon(i)}$ from some $i\geq0$ and $pH^{-1}h_1^{-1}(f^{-1}(W_a))$ is contained in $B_{\delta(m)}$ from $m\in M$ (recall that $p: M\times I\to M$ is projection), or

$(\mathrm{iii})_2$ $H^{-1}h_1^{-1}(f^{-1}(W_a))\subset\Gamma_{\varepsilon(1),\varepsilon(0)}$.

Choosing $\varepsilon(1)$. The set $Z_0=\{a\in X: a$ does not have a neighborhood W_a such that $pH^{-1}h_1^{-1}(f^{-1}(W_a))$ is contained in $B_{\delta(m)}(m)$ for some $m\in M\}$ is closed in X and contained in N_f. Thus, there is a closed f-saturated neighborhood N_0 of $f^{-1}(Z_0)$ missing $h_1H(M\times\{0\})$. Choose $\varepsilon(1)$ so that $\Gamma(0,\varepsilon_1)$ is disjoint from $H^{-1}h_1^{-1}(\mathrm{Int}\,N_0)$ (where '0' denotes the constant map $M\to\{0\}$). Notice that points in Z_0 have neighborhoods satisfying $(\mathrm{iii})_2$.

Choosing $\varepsilon(i+1)$ *recursively.* The set $Z_i=\{a\in X: a$ does not have a neighborhood W_a such that $H^{-1}h_1^{-1}(f^{-1}(W_a))\subset\Gamma(0,\varepsilon(i))\}$ is closed in X and $f^{-1}(Z_i)$ is disjoint from $h_1H(M\times\{0\})$. Choose any neighborhood N of $h_1H(M\times\{0\})$ missing a f-saturated neighborhood of $f^{-1}(Z_i)$ and any $\varepsilon(i+1)$ with $\varepsilon(i+1)(m)<\varepsilon(i)(m)$ for $m\in M$ and with $\Gamma(0,\varepsilon(i+1))\subset H^{-1}h_1^{-1}(\mathrm{Int}\,N)$.

Making a further restriction that $\varepsilon(i)(m)<1/i$, $i=1,2,\dots,$ we specify a homeomorphism $h_0: M\times I\to M\times I$ that is the identity on $M\times\{0,1\}$ and sends the graph of $\varepsilon(i)$ onto the graph of $(1-\delta)^i$ (where $(1-\delta)^i(m)=(1-\delta(m))^i$), $i=1,2,\dots.$

Setting $h_3=h_0H^{-1}$ and $h=h_2h_3h_1^{-1}$, we claim that h is the sought after 'shrinking' homeomorphism. For, if $a\in N_f$ and $(\mathrm{iii})_1$ is satisfied, then $h_0H^{-1}h_1^{-1}(f^{-1}(W_a))\subset\Gamma((1-\delta)^{i+2},(1-\delta)^i)$ and, hence $h_0H^{-1}h_1^{-1}(f^{-1}(W_a))$ is contained in $B_{2\delta(m)}(m)\times B_{2\delta(m)}(t)$ for some $(m,t)\in M\times I$ and condition (ii) reveals that $h(f^{-1}(W_a))$ is $\mathrm{st}^4(\mathcal{V})$-small. If $a\in N_f$ and $(\mathrm{iii})_2$ is satisfied, then $h_0H^{-1}h_1^{-1}(f^{-1}(W_\alpha))\subset\Gamma(1-\delta,1)$ and condition (i) reveals that $h(f^{-1}(W_a))$ is $\mathrm{st}^4(\mathcal{V})$-small. □

Proof of Theorem 3.1. The 'trick' is to approximate f by a fine homotopy equivalence g that satisfies the hypotheses of Proposition 3.2, thereby, discovering that g and, hence, f is a near homeomorphism.

As in the proof of Theorem 2.1, g is the limit of a sequence $f_0, f_1,\dots$ of fine homotopy equivalences specified recursively. For a countable set of embeddings

$e_1, e_2, \ldots$ of the Hilbert cube I^∞ into M^s chosen as in the proof of Theorem 2.1, the f_i's are chosen as in the proof of Theorem 2.1 so that $f_0 = f$, f_i is one to one over the image of $\bigcup\{e_k(I^\infty)\colon 1 \leq k \leq i\}$, and $f_i = f_{i-1}$ on $\bigcup\{e_k(I^\infty)\colon 1 \leq k \leq i-1$.

4. Characterization of s-manifolds

We are now prepared to present a proof of Torunczyk's characterization of s-manifolds that is based on a 'categorical' application of Theorems 2.1 and 3.1.

Characterization Theorem. *A topologically complete ANR X is an s-manifold if and only if X satisfies the discrete approximation property.*

Proof. Name a fine homotopy equivalence $f\colon M^s \to X$ from an *s-manifold to X*. Perhaps the easiest method for producing such a resolution is due to Miller [15]. While the setting there is that of locally compact ANR's the same techniques apply to produce a fine homotopy equivalence $f_0\colon M^s \to X \times [0, 1)$ and f is obtained by composing f_0 with the projection onto X. (The reader is referred to [13; Appendix 2] for a further discussion of adjusting Miller's approach to the setting of s-manifolds.)

Name a countable family of embeddings $e_1, e_2, \ldots$ of the Hilbert cube I^∞ into X, having pairwise disjoint images, that is dense in the space of maps of I^∞ to X. Adopting the notation of Section 1, f factors as $f = p_1 q$ where $q\colon M \to M \cup_f e_1(I^\infty)$ and $p_1\colon M \cup_f e_1(I^\infty) \to X$ are the 'induced' maps. Now, $N_q \subset e_1(I^\infty)$ and $e_1(I^\infty)$ is a strong Z-set in $M \cup_f e_1(I^\infty)$, the latter is essentially a consequence of $e_1(I^\infty)$ being a strong Z-set in X (see Proposition 1.1). Theorem 2.1 applies to yield a homeomorphism h approximating q. Set $f_i = p_1 h$ and observe that f_1 approximates f and is one to one over $e_1(I^\infty)$. Successively, repeat this process thereby producing a sequence $f = f_0, f_1, f_2, \ldots$ of fine homotopy equivalences such that f_i approximates f_{i-1} and f_i is one to one over $\bigcup\{e_k(I^\infty)\colon 1 \leq k \leq i\}$. Exercising adequate care in specifying 'closeness' leads to the f_i's converging to a fine homotopy equivalence g approximating f that is one to one over $\bigcup\{e_k(I^\infty)\colon k \geq 1\}$. Theorem 3.1 applies to show that g is a near homeomorphism. □

The principal facets of s-manifolds appearing in the proof of the Characterization Theorem are Z-set unknotting (controlled version) and projection $M^s \times I \to M$ being a near homeomorphism. The former fits comfortably under the heading 'geometrical property of s-manifolds' while the later is a hybrid property, momentarily removing us from the world of s-manifolds (why is $M \times I$ an s-manifold?). In the Appendix, an argument is outlined that establishes that $M^s \times I$ is an s-manifold and that projection $M^s \times I \to M$ is a near homeomorphism.

The following are consequences of the Characterization Theorem (and its proof).

Corollary 4.1 [18]. *If X is a topologically complete ANR, then $X \times s$ is an s-manifold.*

Corollary 4.2 [14, 21]. *A fine homotopy equivalence $f: M^s \to N^s$ between s-manifolds is a near homeomorphism.*

Corollary 4.3 [2]. *Hilbert space l_2 is homeomorphic to $s = (0, 1)^\infty$.*

Corollary 4.4 [11]. *If A is a σ-Z-set in an s-manifold M^s, then the inclusion $M - A \to M$ is a near homeomorphism.*

The proof of the first corollary relies on verifying that $X \times s$ satisfies the discrete approximation property (which is left to the reader), while that of the second is a consequence of the proof of the Characterization Theorem. The third follows from the first two since both projections $l_2 \times s \to l_2$ and $l_2 \times s \to s$ are fine homotopy equivalences, $l_2 \times s$ is an s-manifold, and l_2 satisfies the discrete approximation property (see [21]). The proof of the fourth rests on showing that $M - A$ satisfies the discrete approximation property (which is left to the reader) and applying Corollary 4.2 to the inclusion $M - A \to M$.

5. Example: Discrete *n*-cells for all *n*

The finite dimensional version of the discrete approximation property has been labeled the *discrete n-cells property*, namely: for each countable family of maps $\alpha_i: I^n \to X$, $i = 1, 2, \ldots$, of n-cells to X and open cover $\mathcal{U}$ of X, there are $\mathcal{U}$-approximations $\beta_i: I^n \to X$, $i = 1, 2, \ldots$, such that the collection $\{\beta_i(I^n)\}_{i \geq 1}$ is discrete in X. The example constructed below, an embellishment of the example presented in Section 1, arises as the reduced product $(K \times s)_{a_0}$ and satisfies the discrete n-cells property for each n but does not satisfy the discrete approximation property. The space K is a topologically complete AR that splits as $(K - a_0) \cup \{a_0\}$ where $\{a_0\}$ is a Z-set and $K - a_0$ is a contractible polyhedron.

The discrete cells properties have been analysed by Bowers [7, 8, 9]. In particular, he has shown that if $X \subset X^*$ where X^* is a locally compact ANR with $X^* - X$ a σ-Z-set in X^*, and X satisfies the discrete n-cells property for each n, then X satisfies the discrete approximation property. Consequently, the examples constructed below and in the next section (as well as that in Section 1) fail to possess 'nice' ANR local compactifications.

Example. We start with the set A, illustrated in Fig. 2, consisting of a null sequence of pairwise disjoint arcs $J_1, J_2, \ldots$ converging to a point a_0. As suggested by the notation B^n is an n-cell. Each B^n intersects only J_{n-1} and J_n, meeting each in an endpoint lying in its boundary.

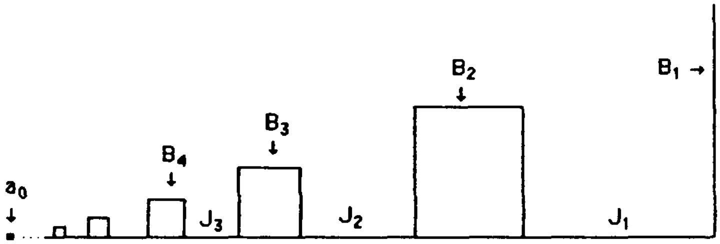


Fig. 2.

Set $\partial A = \bigcup\{\partial B^n\colon n \geq 1\}$, $J = \bigcup\{J_n\colon n \geq 1\}$, and $K = (A \times \{0\}) \cup (\partial A \times [0, 1]) \cup (J \times [0, 1])$, the latter being viewed as a subset of $A \times [0, 1]$. There is an instantaneous deformation of K into $K - \{a_0\}$, similar to that described for the example in Section 1, that shows that a_0 is a Z-set in K. However, a_0 is not a strong Z-set. If it were, there would be a neighborhood N of a_0 such that each of the components of $\partial A \times \{1\}$ would lie in $K - N$ and would be contractible in $K - N$. This is not possible since the contraction of $\partial B^n \times \{1\}$ must 'cover' $B^n \times \{0\}$ (since $K - \{x\}$ retracts to $\partial B^n \times \{1\}$ for any $x \in \operatorname{Int}(B^n \times \{0\})$). In order to show that $(K \times s)_{a_0}$ satisfies the discrete n-cells property for each n, it suffices to show that, given an open cover $\mathcal{U}$ of K and an integer n, any countable family of maps $\alpha_1, \alpha_2, \ldots$ of the n-cell I^n to K have a $\mathcal{U}$-approximations $\beta_1, \beta_2, \ldots$ whose images simultaneously miss a neighborhood of a_0. For an integer m, determine subsets of A by setting $A(m) = \{a_0\} \cup (\bigcup\{B^i\colon i \geq m\}) \cup (\{J_i\colon i \geq m\})$, $\partial A(m) = \bigcup\{\partial B^i\colon i \geq m\}$, and $J(m) = \{J_i\colon i \geq m\}$. Choose m and $\varepsilon > 0$ so that $N = (A(m) \times \{0\}) \cup (\partial A(m) \times [0, \varepsilon]) \cup (J(m) \times [0, \varepsilon])$ is contained in an element of $\mathcal{U}$. Assuming further that $m > n$, it is possible to adjust each α_i so that $\alpha_i(I^n) \cap [\{a_0\} \cup (\bigcup\{B^i - \partial B^i\colon i \geq m\})] = \emptyset$ and, then choose β_i, $\mathcal{U}$-close to α_i, so that the image of β_i misses

$$(A(m+1) \times \{0\}) \cup (\partial A(m+1) \times [0, \varepsilon/2]) \cup (J(m+1) \times [0, \varepsilon/2]).$$

6. Example: Discrete carriers

A homological analysis using homology carriers appeared in [12] that lead to a characterization of those Hilbert cube manifold factors such that $X \times I^n$ for some n (or, more generally, $X \times Y$ for some finite dimensional Y) is a Hilbert cube manifold. The interval I and its multiple self products I^n cannot play a comparable role for s-manifolds since the product $X \times Y$, for Y locally compact, satisfies the discrete approximation property if and only if X itself satisfies the same property. In [7, 8, 9], Bowers established that, for s-manifolds, the appropriate analogue of the interval is any nowhere locally compact 1-dimensional AR, say A, that arises as $A = A^* - E$ where A^* is a dendrite whose endpoints are dense and E is a dense σ-compact subset of the endpoints.

The efforts in [12] established that the disjoint carriers property (described below) is characteristic of locally compact ANR's X such that $X \times I^2$ is a Hilbert cube manifold and that $X \times Y$ a Hilbert cube manifold for Y finite dimensional implies that X satisfies the disjoint carriers property. The latter property states that, for each pair of integers $q(1)$, $q(2)$, each pair of open pairs (U_1, V_1), (U_2, V_2), and each pair of homology elements $z_i \in H_{q(i)}(U_i, V_i; Z)$ for $i = 1, 2$, there are disjoint compact pairs (C_1, D_1), (C_2, D_2) such that, for $i = 1, 2$, $(C_i, D_i) \subset (U_i, V_i)$ and $z_i \in \mathrm{Im}\{H_{q(i)}(C_i, D_i; Z) \to H_{q(i)}(U_i, V_i; Z)\}$. The strategy in [12] is to prove a 'Hurewicz Theorem'; the disjoint carriers property together with the disjoint 2-cells property implies the disjoint n-cells property for all n and, then, to apply Torunczyk's characterization of Hilbert cube manifolds [20].

While the discrete carriers property (described below) together with the discrete 2-cells property implies the discrete n-cells property for each n, see [7], the example presented in this section satisfies the discrete carriers property and the discrete n-cells property for each n, but not the discrete approximation property.

The *discrete carriers property* is that for each open cover $\mathcal{U}$ of X and sequence of homology elements $\{z_i \in H_{q(i)}(U_i, V_i; Z): i \geq 1\}$ where the (U_i, V_i)'s are open pairs in X, there are compact pairs $(C_i, D_i) \subset (\mathrm{st}(U_i, \mathcal{U}), \mathrm{st}(V_i, \mathcal{U}))$ such that the family $\{C_i: i \geq 1\}$ is discrete and the image of z_i in $H_{q(i)}(\mathrm{st}(U_i, \mathcal{U}), \mathrm{st}(V_i, \mathcal{U}); Z)$ is contained in the image of $H_{q(i)}(C_i, D_i; Z)$.

Example. As has been the case with the earlier examples, the example has the form $(L \times s)_{a_0}$ where $L - \{a_0\}$ is a polyhedron and a_0 is a Z-set in L. Furthermore, for each n, any sequence of maps $\alpha_1, \alpha_2, \ldots$ of the n-cell I^n to L can be moved by small moves off a neighborhood of a_0, as can countably many homology elements. These last two properties insure that $(L \times s)_{a_0}$ satisfies the discrete n-cells property for each n and the discrete carriers property. Since a_0 is not a strong Z-set in $(L \times s)_{a_0}$, the latter cannot satisfy the discrete approximation property.

The building blocks for the example comprise a sequence of PL-maps $\{f_n : (K_n, w_n) \to (K_{n-1}, w_{n-1}): n \geq 0\}$ between compact connected and simply connected polyhedra such that:

(a) $K_0 = \{w_0\}$ and every finite composition $f_2 f_3 \cdots f_n$ is essential;

(b) for each n, $(f_n)_{\#} : \pi_i(K_n, w_n) \to \pi_i(K_{n-1}, w_{n-1})$ is the zero homomorphism for $i \leq n$; and

(c) for each n, $(f_n)_* : H_*(K_n; Z) \to H_*(K_{n-1}; Z)$ is the zero homomorphism (in all positive dimensions).

Such a sequence of maps and polyhedra can be concocted using the examples in [1] or [17].

The example is constructed starting with the planar first quadrant $Q = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$. Form an adjunction space Q_1 from $Q \cup [\bigcup \{K_{m-n} \times (n, m)$: m, n are integers with $0 \leq n \leq m\}]$ by identifying the point $w_{m-n} \times (n, m)$ with the integer lattice point (n, m). Form an adjunction space Q_2 by attaching, for each pair of integers $0 \leq n \leq m$, a copy of the mapping cylinder of $f_{m-n} : K_{m-n} \to K_{m-n-1}$

to Q_1, identifying the 'top' of the mapping cylinder with $K_{m-n}\times(n, m)$, the 'base' with $K_{m-n-1}\times(n+1, m)$, and the ray from w_{m-n} to w_{m-n-1} to the horizontal ray from (n, m) to $(n+1, m)$. (The space Q_2 is the Q with a copy of the mapping telescope of $K_m\to K_{m-1}\to\cdots\to K_1\to K_0$ attached along the horizontal line $y=m$ by glueing the ray in the telescope from w_m to w_0 onto the ray in Q from $(0, m)$ to (m, m).) Finally, set $L=Q_2\cup\{a_0\}$ where a closed neighborhood base $\{B_i: i\geq 0\}$ at a_0 is determined as follows. For an integer $i\geq 0$, B_i consists of all points in Q on or to the right of the vertical line $x=i$ together with those mapping cylinders attached to rays in this region.

Since $K_0=\{w_0\}$, the part of Q on or below the diagonal $x=y$, named Q_-, is embedded in L. (In fact, $Q_-\cup\{a_0\}$ is a 2-cell.) In particular, L is contractible, the contraction being comprised of a 'purely horizontal' move to the right pushing L into the contractible space $Q_-\cup\{a_0\}$. Similarly, each B_i is contractible and it follows easily that L is an AR. The space L is topologically complete since it is the union of a topologically complete space Q_2 and a point a_0. A not too difficult variation of the description of the homotopies just given leads to the construction of an instantaneous deformation of L into $L-\{a_0\}$, the details are omitted. (The first property of L established below is sufficient to show that $\{a_0\}$ is a Z-set and, thus, establish the existence of such a deformation.)

Observe that $\{a_0\}$ is not a strong Z-set, for if it were, there would be a neighborhood N of a_0 with $K_m\times(0, m)\subset L-N$ and with this inclusion being null homotopic in $L-N$, for each $m\geq 0$. It follows from property (a) of the sequence $\{f_n: K_n\to K_{n-1}\}$ that, if $B_i\subset N$ and $m\geq i$, then the inclusion $K_m\times(0, m)\to L-N$ is not null homotopic.

First Property of L. *For each $k\geq 0$, maps of the i-sphere into* $\mathrm{Fr}(B_k)$ *are null homotopic in* $\mathrm{cl}(B_k-B_{k+n})$ *for $i\leq n$.*

Second Property of L. *For each $k\geq 0$, the inclusion induces the zero homomorphism* $H_*(\mathrm{Fr}(B_k); Z)\to H_*(\mathrm{cl}(B_k-B_{k+1}); Z)$ *in all positive dimensions.*

The properties are fairly direct consequences of the properties (b) and (c) that the sequence $\{f_n: K_n\to K_{n-1}\}$ satisfies, details are left to the interested reader. Finally, the properties for L claimed in the first paragraph of this description follow easily and we conclude that $(L\times s)_{a_0}$ satisfies the discrete n-cells property for each n and the discrete carriers property but is not an s-manifold since a_0 is not a strong Z-set.

7. Incomplete manifolds σ and Σ

The spaces σ and Σ described in the introduction can also be determined as subspaces of Hilbert space l_2. Namely, $\sigma=\{(x_i): x_i=0$ for all but finitely many $i\}$ and $\Sigma=\{(x_i): \sum(ix_i)^2<\infty\}$. The characterization of these appearing in [16] is incomplete for its proofs implicity use that Z-sets are strong Z-sets. Unfortunately, this is not a consequence of the strong embedding properties that comprise the hypotheses, the spaces $(C\times\sigma)_{a_0}$ and $(C\times\Sigma)_{a_0}$ being counterexamples, where C is the example of Section 1. Details of this adjustment will appear in a forthcoming paper of the third author.

Appendix

While the proofs in the paper rely heavily on a variety of basic results and techniques that are found in one form or other in papers that focus on characterizing manifolds, there are two crucial features of s-manifolds that play critical roles. The first is the controlled version of Z-set unknotting (see [3]) and the second is the fact that projection $M^s\times I\to M^s$ is a near homeomorphism (see [5]). The first of these fits easily under the heading 'geometrical property of s-manifolds' and we are content to state a version below that is sufficient for our purposes. The second is a 'hybrid' result since it involves first recognizing that $M^s\times I$ is an s-manifold (the central topic of this exposition) and, then, establishing that the projection is a near homeomorphism. We will say more about this below.

Z-set unknotting

Given a Z-embedding $F: A\times I\to M^s$ into an s-manifold (i.e., a closed embedding onto a Z-set), there is an isotopy $\{h_t: 0\leq t\leq 1\}$ of M such that h_0 is the identity, $h_tF|_{A\times\{0\}}=F|_{A\times\{t\}}$, and outside any prechosen neighborhood of the image of F, each h_t is the identity. Furthermore, if the 'tracks' $\{F(a\times I): a\in A\}$ refine an open cover $\mathcal{W}$, then we can require the 'tracks' of the isotopy refine the cover also.

Projection $M\times I\to M$ is a near homeomorphism

First, we limit ourselves to the special case of the projection $s\times I\to s$ (a central theme in [5] involves 'piecing together' local data to produce global information). A relevant illustration of this parenthetical remark is that, while it is obvious that $s\times(0,1)$ and $s\times s$ are homeomorphic to s, there is work involved in showing that the projections $s\times(0,1)\to s$ and $s\times s\to s$ (more generally, $M^s\times(0,1)\to M^s$ and $M^s\times s\to M^s$) are near homeomorphisms. The reader is referred to [5] for the details.

It should be evident that, in the presence of $s\times(0,1)\to s$ being a near homeomorphism, the projection $s\times I\to s$ is a near homeomorphism provided the inclusion $s\times(0,1)\subset s\times I$ is a near homeomorphism. One way to establish this is to use the 'uniqueness' of *compact absorption sets* [11]. The point is that both $I^\infty\times I-s\times(0,1)$ and $I^\infty\times I-s\times I$ are compact absorption sets and, hence, there is a homeomorphism $h: I^\infty\times I\to I^\infty\times I$ 'near' the identity that takes one of these sets onto the other. Consequently, the restriction of h is a homeomorphism between $s\times(0,1)$ and $s\times I$. (The reader can legitimately object that the homeomorphisms produced this way are not sufficient to establish that the inclusion $s\times(0,1)\subset s\times I$ is a near homeomorphism, since the control is an open cover of $I^\infty\times I$ not of $s\times I$. The extra control is gained by, given an open cover $\mathcal{W}$ of $s \times I$, expanding this to a collection $\mathcal{W}'$ of open subsets of $I^\infty \times I$ with $\mathcal{W} = \{W \cap s \times I \colon W \in \mathcal{W}'\}$ and applying the above argument to the I^∞-manifold $\bigcup \{W \in \mathcal{W}'\}$.)

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